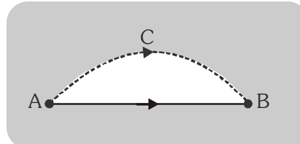
KUSAL ACADEMY

KINEMATICS

- **Distance and Displacement**

Total length of path (ACB) covered by the particle, in definite time interval is called distance. Displacement vector or displacement is the minimum distance (AB) and directed from initial position to final position.

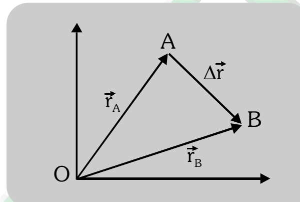


- **Displacement in terms of position vector**

From ΔOAB $\qquad \Delta\vec{r} = \vec{r}_B - \vec{r}_A$

$\vec{r}_B = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ and $\vec{r}_A = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$

$\Delta\vec{r} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$



- **Average velocity** $\quad \dfrac{\text{Displacement}}{\text{Time interval}} = \vec{v}_{av} = \dfrac{\Delta\vec{r}}{\Delta t}$

- **Average speed** $\quad \dfrac{\text{Distance travelled}}{\text{Time interval}}$

- **For uniform motion** $\quad$ Average speed = | average velocity | = | instataneous velocity |

- **Velocity** $\quad \vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{d}{dt}\left(x\hat{i} + y\hat{j} + z\hat{k}\right) = \dfrac{dx}{dt}\hat{i} + \dfrac{dy}{dt}\hat{j} + \dfrac{dz}{dt}\hat{k} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$

- **Average Acceleration** $\quad \dfrac{\text{change in velocity}}{\text{total time taken}} = \vec{a}_{av} = \dfrac{\Delta\vec{v}}{\Delta t}$

- **Acceleration** $\quad \vec{a} = \dfrac{d\vec{v}}{dt} = \dfrac{d}{dt}\left(v_x\hat{i} + v_y\hat{j} + v_z\hat{k}\right) = \dfrac{dv_x}{dt}\hat{i} + \dfrac{dv_y}{dt}\hat{j} + \dfrac{dv_z}{dt}\hat{k} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

Important points about 1D motion

- Distance ≥ | displacement | and Average speed ≥ | average velocity |
- If distance > | displacement | this implies

  (a) atleast at one point in path, velocity is zero.

  (b) The body must have retarded during the motion
- Speed increase if acceleration and velocity both are positive or negative

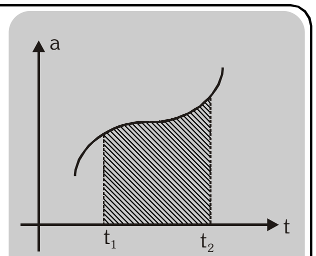
  (i.e. both have same sign)

- **In 1-D motion** $a = \dfrac{dv}{dt} = v\dfrac{dv}{dx}$
- **Graphical integration in Motion analysis**

If a = f(t)

$$a = \frac{dv}{dt} \Rightarrow \int_{v_1}^{v_2} dv = \int_{t_1}^{t_2} a\,dt \Rightarrow v_2 - v_1 = \int_{t_1}^{t_2} a\,dt$$

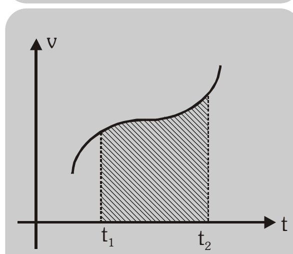
⇒ Change in velocity = Area between acceleration curve and time axis, from t_1 to t_2



shaded area = change in velocity

If v = f(t)

$$v = \frac{dx}{dt} \Rightarrow \int_{x_1}^{x_2} dx = \int_{t_1}^{t_2} v\,dt \Rightarrow x_2 - x_1 = \int_{t_1}^{t_2} v\,dt$$



shaded area = displacement

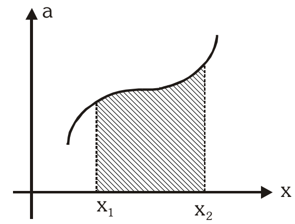
$\Rightarrow$ Change in position = displacement

= area between velocity curve and time axis, from t_1 to t_2.

If $a = f(x)$

$$a = \frac{vdv}{dx} \Rightarrow \int_{v_i}^{v_f} vdv = \int_{x_i}^{x_f} adx$$

$$\frac{v_f^2 - v_i^2}{2} = \text{Area under a-x curve}$$



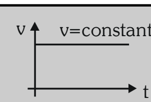
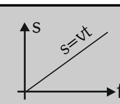
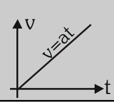
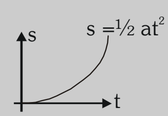
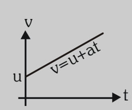
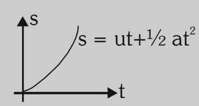
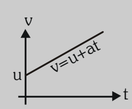
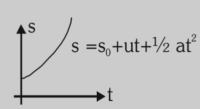
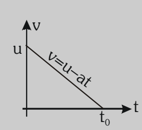
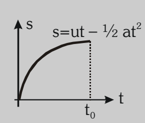
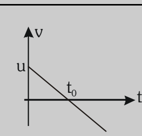
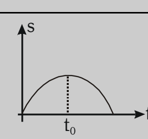
shaded area = $\frac{v_2^2 - v_1^2}{2}$

Important point about graphical analysis of motion

- Instantaneous velocity is the slope of position time curve $\left(v = \frac{dx}{dt}\right)$
- Slope of velocity-time curve = instantaneous acceleration $\left(a = \frac{dv}{dt}\right)$
- v-t curve area gives displacement. $\left[\Delta x = \int vdt\right]$
- a-t curve area gives change in velocity. $\left[\Delta v = \int adt\right]$

Displacement —Differentiation→ **Velocity** —Differentiation→ **Acceleration**

Displacement ←Integration— **Velocity** ←Integration— **Acceleration**

	Different Cases	**v–t graph**	**s–t graph**
1.	Uniform motion		
2.	Uniformly accelerated motion with $u = 0$ at $t = 0$		
3.	Uniformly accelerated with $u \neq 0$ at $t = 0$		
4.	Uniformly accelerated motion with $u \neq 0$ and $s = s_0$ at $t = 0$		
5.	Uniformly retarded motion till velocity becomes zero		
6.	Uniformly retarded then accelerated in opposite direction		

Motion with constant acceleration : Equations of motion		
In vector form :	$\vec{v} = \vec{u} + \vec{a}t$	$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 = \vec{s} = \left(\frac{\vec{u}+\vec{v}}{2}\right)t = \vec{u}t + \frac{1}{2}\vec{a}t^2 = \vec{v}t - \frac{1}{2}\vec{a}t^2$
	$v^2 = u^2 + 2\vec{a}.\vec{s}$	$\vec{s}_{n^{th}} = \vec{u} + \frac{\vec{a}}{2}(2n-1)$
[$S_{n^{th}} \rightarrow$ displacement in n^{th} second]		
In scalar form	$v = u + at$	$s = \left(\frac{u+v}{2}\right)t = ut + \frac{1}{2}at^2 = vt - \frac{1}{2}at^2$
(for one dimensional motion) :	$v^2 = u^2 + 2as$	$s_{n^{th}} = u + \frac{a}{2}(2n-1)$

RELATIVE MOTION

There is no meaning of motion without reference or observer. If reference is not mentioned then we take the ground as a reference of motion. Generally velocity or displacement of the particle w.r.t. ground is called actual velocity or actual displacement of the body. If we describe the motion of a particle w.r.t. and object which is also moving w.r.t. ground then velocity of particle w.r.t. ground is its actual velocity $(\vec{v}_{act})$ and velocity of particle w.r.t. moving object is its relative velocity $(\vec{v}_{rel.})$ and the velocity of moving object (w.r.t. ground) is the reference velocity $(\vec{v}_{ref.})$ then $\vec{v}_{rel} = \vec{v}_{act} - \vec{v}_{ref}$

$$\boxed{\vec{v}_{actual} = \vec{v}_{relative} + \vec{v}_{reference}} \xrightarrow{\text{Differentiation}} \boxed{\vec{a}_{actual} = \vec{a}_{relative} + \vec{a}_{reference}}$$

If $\vec{a}_{rel} = 0$

$\Rightarrow \vec{v}_{rel} = \text{constant}$

then $\Rightarrow \vec{S}_{rel} = \vec{v}_{rel} \times \text{time}$

If $\vec{a}_{rel} = \text{constant}$

then we can we equation of metion in relative form

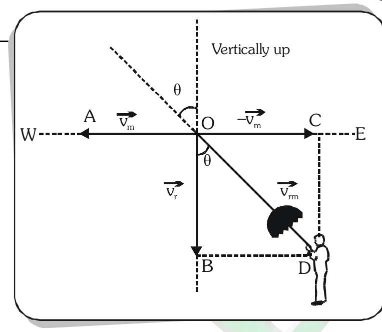
$$\vec{v}_{rel} = \vec{u}_{rel} + \vec{a}_{rel}t \quad \text{... (i)}$$

$$\vec{s}_{rel} = \vec{u}_{rel}t + \frac{1}{2}\vec{a}_{rel}t^2 \quad \text{... (ii)}$$

$$\vec{v}_{rel} \cdot \vec{v}_{rel} = \vec{u}_{rel} \cdot \vec{u}_{rel} + 2\left(\vec{a}_{rel} \cdot \vec{s}_{rel}\right)$$

Relative velocity of Rain w.r.t. the Moving Man :

A man walking west with velocity $\vec{v}_m$, represented by $\overrightarrow{OA}$. Let the rain be falling vertically downwards with velocity $\vec{v}_r$, represented by $\overrightarrow{OB}$ as shown in figure.



The relative velocity of rain w.r.t. man $\vec{v}_{rm} = \vec{v}_r - \vec{v}_m$ will be represented by diagonal $\overrightarrow{OD}$ of rectangle OBDC.

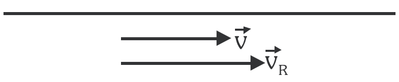
$$\therefore v_{rm} = \sqrt{v_r^2 + v_m^2 + 2v_r v_m \cos 90^\circ} = \sqrt{v_r^2 + v_m^2}$$

If θ is the angle which $\vec{v}_{rm}$ makes with the vertical direction then $\tan\theta = \frac{BD}{OB} = \frac{v_m}{v_r} \Rightarrow \theta = \tan^{-1}\left(\frac{v_m}{v_r}\right)$

Swimming into the River :

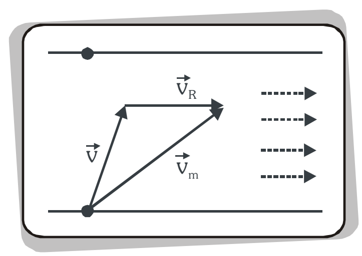
A man can swim with velocity $\vec{v}$, i.e. it is the velocity of man w.r.t. still water. If water is also flowing with velocity $\vec{v}_R$ then velocity of man relative to ground $\vec{v}_m = \vec{v} + \vec{v}_R$

- If the swimming is in the direction of flow of water or along the downstream then $v_m = v + v_R$

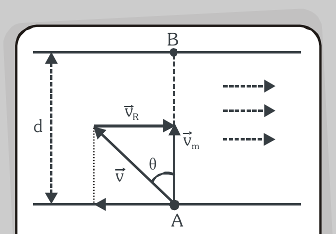


- If the swimming is in the direction opposite to the flow of water or along the upstream then $v_m = v - v_R$

- If man is crossing the river as shown in the figure i.e. $\vec{v}$ and $\vec{v}_R$ not collinear then use the vector algebra $\vec{v}_m = \vec{v} + \vec{v}_R$ **(assuming v > v_R)**



For shortest path

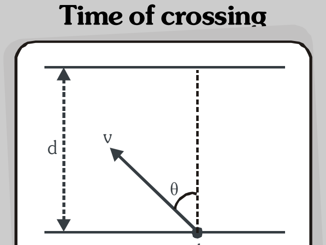


For minimum displacement

To reach at B:

$v\sin\theta = v_R \Rightarrow \sin\theta = \frac{v_R}{v}$

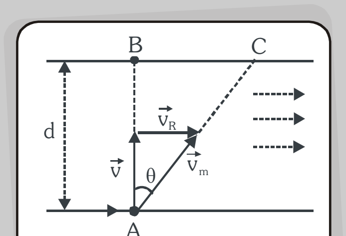
Time of crossing



$t = \frac{d}{v\cos\theta}$

Note : If $v_R > v$ then for minimum drifting $\sin\theta = \frac{v}{v_R}$

For minimum time



(for minimum time)

then $t_{min} = \frac{d}{v}$

MOTION UNDER GRAVITY

If a body is thrown vertically up with a velocity u in the uniform gravitational field (neglecting air resistance) then

(i) Maximum height attained $H = \frac{u^2}{2g}$

(ii) Time of ascent = time of descent = $\frac{u}{g}$

(iii) Total time of flight = $\frac{2u}{g}$

(iv) Velocity of fall at the point of projection = u (downwards)

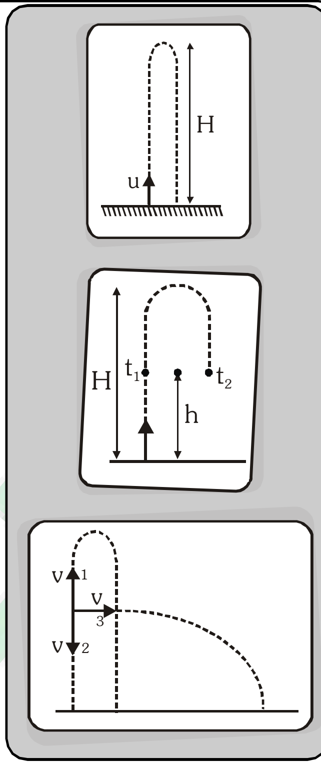
(v) **Gallileo's law of odd numbers** : For a freely falling body ratio of successive distance covered in equal time internval 't'

$S_1 : S_2 : S_3 : \ldots S_n = 1 : 3 : 5 : \ldots : 2n-1$

At any point on its path the body will have same speed for upward journey and downward journey. If a body thrown upwards crosses a point in time t_1 & t_2 respectively then height of point $h = \frac{1}{2} g t_1 t_2$

Maximum height $H = \frac{1}{8} g(t_1 + t_2)^2$

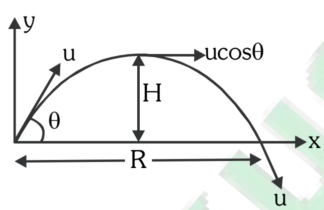
A body is thrown upward, downward & horizontally with same speed takes time t_1, t_2 & t_3 respectively to reach the ground then $t_3 = \sqrt{t_1 t_2}$ & height from where the particle was throw is $H = \frac{1}{2} g t_1 t_2$



PROJECTILE MOTION

Horizontal Motion

$u\cos\theta = u_x$; $a_x = 0$; $x = u_x t = (u\cos\theta)t$



Vertical Motion

$v_y = u_y - gt$ where $u_y = u\sin\theta$;

$y = u_y t - \frac{1}{2}gt^2 = u\sin\theta t - \frac{1}{2}gt^2$

Net acceleration = $\vec{a} = a_x\hat{i} + a_y\hat{j} = -g\hat{j}$

At any instant :

$v_x = u\cos\theta$, $v_y = u\sin\theta - gt$

Velocity of particle at time t :

$\vec{v} = v_x\hat{i} + v_y\hat{j} = u_x\hat{i} + (u_y - gt)\hat{j} = u\cos\theta\hat{i} + (u\sin\theta - gt)\hat{j}$

If angle of velocity $\vec{v}$ from horizontal is α, then

$$\tan\alpha = \frac{v_y}{v_x} = \frac{u_y - gt}{u_x} = \frac{u\sin\theta - gt}{u\cos\theta} = \tan\theta - \frac{gt}{u\cos\theta}$$

- At highest point : $v_y = 0$, $v_x = u\cos\theta$
- Time of flight : $T = \frac{2u_y}{g} = \frac{2u\sin\theta}{g}$
- Horizontal range :

$$R = (u\cos\theta)T = \frac{2u^2\sin\theta\cos\theta}{g} = \frac{u^2\sin 2\theta}{g} = \frac{2u_x u_y}{g}$$

It is same for θ and (90°–θ) and maximum for θ = 45°

- Maximum height : $H = \frac{u_y^2}{2g} = \frac{u^2\sin^2\theta}{2g} = \frac{1}{8}gT^2$
- $\frac{H}{R} = \frac{1}{4}\tan\theta$
- Equation of trajectory

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta} = x\tan\theta\left(1 - \frac{x}{R}\right)$$

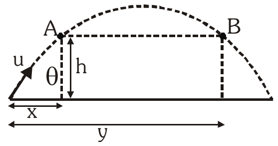
For projectile motion :

A body crosses two points at same height in time t_1 and t_2 the points are at distance x and y from starting point then

(a) $x + y = R$ (b) $t_1 + t_2 = T$

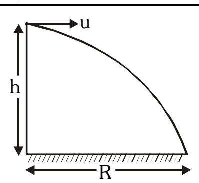
(c) $h = \frac{1}{2}\, g t_1 t_2$

(d) Average velocity from A to B is $u\cos\theta$



Note :- If a person can throw a ball to a maximum distance 'x' then the maximum height to which he can throw the ball will be (x/2)

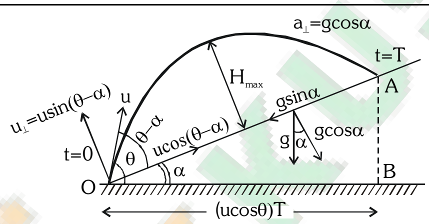
Horizontal projection from some height



- Time of flight $T = \sqrt{\frac{2h}{g}}$
- Horizontal range $R = uT = u\sqrt{\frac{2h}{g}}$
- Angle of velocity at any instant with horizontal

$$\theta = \tan^{-1}\left(\frac{gt}{u}\right)$$

Projectile motion on inclined plane- up motion

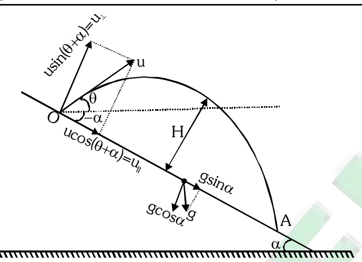


- Time of flight: $T = \frac{2u_\perp}{g_\perp} = \frac{2u\sin(\theta-\alpha)}{g\cos\alpha}$
- Maximum height : $H_{max} = \frac{u_\perp^2}{2g_\perp} = \frac{u^2\sin^2(\theta-\alpha)}{2g\cos\alpha}$
- Range on inclined plane :

$$R = OA = \frac{2u^2\sin(\theta-\alpha)\cos\theta}{g\cos^2\alpha}$$

- Max. range : $R_{max} = \frac{u^2}{g(1+\sin\alpha)}$ at angle $\theta = \frac{\pi}{4} + \frac{\alpha}{2}$

Projectile motion on inclined plane – down motion (put α = −α in above)



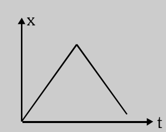
- Time of flight : $T = 2t_H = \frac{2u_\perp}{a_\perp} = \frac{2u\sin(\theta+\alpha)}{g\cos\alpha}$
- Maximum height : $H = \frac{u_\perp^2}{2a_\perp} = \frac{u^2\sin^2(\theta+\alpha)}{2g\cos\alpha}$
- Range on inclined plane :

$$R = OA = \frac{2u^2\cos\theta\sin(\theta+\alpha)}{g\cos^2\alpha}$$

- Max. range: $R_{max} = \frac{u^2}{g(1-\sin\alpha)}$ at angle $\theta = \frac{\pi}{4} - \frac{\alpha}{2}$

KEY POINTS :

- A positive acceleration can be associated with a "slowing down" of the body because the origin and the positive direction of motion are a matter of choice.
- The x-t graph for a particle undergoing rectilinear motion, cannot be as shown in figure because infinitesimal changes in velocity are physically possible only in infinitesimal time.



- In oblique projection of a projectile the speed gradually decreases up to the highest point and then increases because the tangential acceleration opposes the motion till the particle reaches the highest point, and then it favours the motion of the particle.
- In free fall, the initial velocity of a body may not be zero.
- A body can have acceleration even if its velocity is zero at an instant.
- Average velocity of a body may be equal to its instantaneous velocity.
- The trajectory of an object moving under constant acceleration can be straight line or parabola.
- The path of one projectile as seen from another projectile is a straight line as relative acceleration of one projectile w.r.t. another projectile is zero.