KUSAL ACADEMY

ROTATIONAL MOTION

RIGID BODY

Rigid body is defined as a system of particles in which distance between each pair of particles remains constant (with respect to time) that means the shape and size do not change, during the motion.

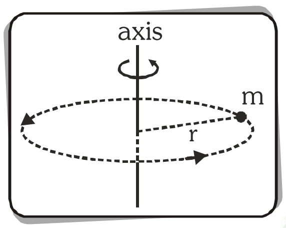
Eg : Fan, Pen, Table, stone and so on.

Type of Motion of rigid body
- Pure Translational Motion
- Pure Rotational Motion
- Combined Translational and Rotational Motion

ROTATIONAL MOTION

Moment of Inertia

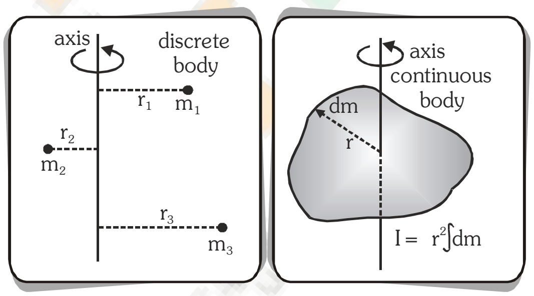
The virtue by which a body revolving about an axis opposes the change in rotational motion is known as moment of inertia.



- The moment of inertia of a particles with respect to an axis of rotation is equal to the product of mass of the particle and square of distance from rotational axis. $I = mr^2$

  r = perpendicular distance from axis of rotation

- Moment of inertia of system of particle



$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + ...$$

For Rigid Bodies :

Moment of inertia of a rigid body about any axis of rotation. $I = \int dm\, r^2$

Radius of Gyration (K)

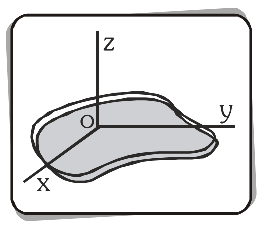
K has no meaning without axis of rotation.

$I = MK^2$ K is a scalar quantity

Radius of gyaration : $K = \sqrt{\frac{I}{M}}$

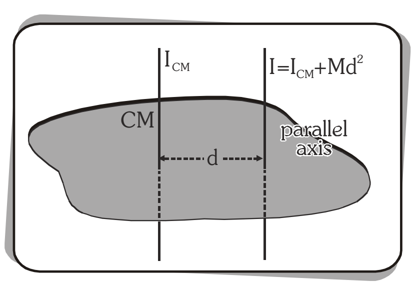
Perpendicular axis Theorems : $I_z = I_x + I_y$

(body lies on the x-y plane)



(Valid only for 2-dimensional body)

Parallel axis Theorem : $I = I_{CM} + Md^2$



(for all type of bodies)

I_{CM} = moment of inertia about the axis Passing through the centre of mass

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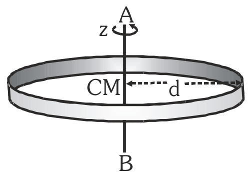
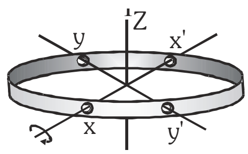
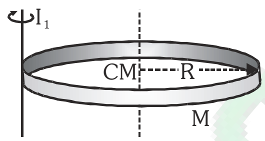
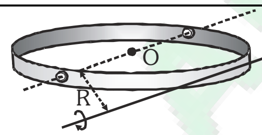
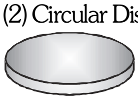
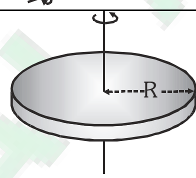
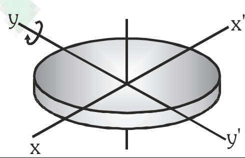
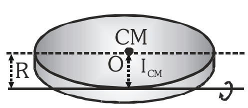
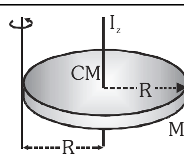
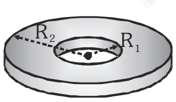
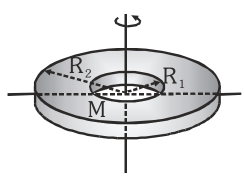
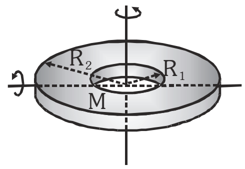
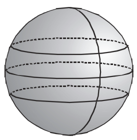
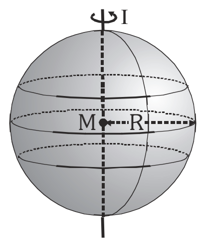
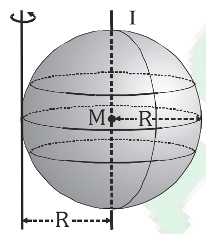
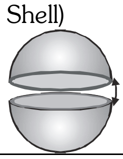
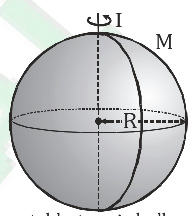
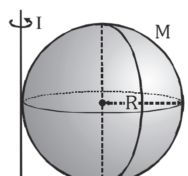
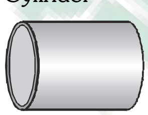
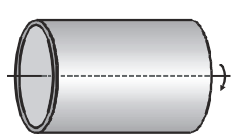
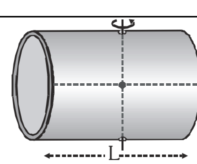
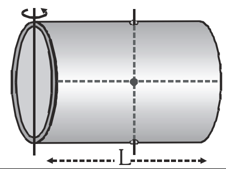
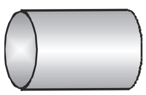
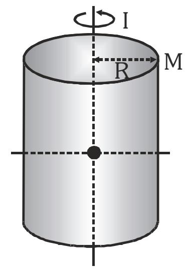
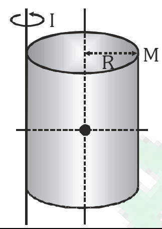
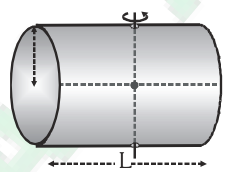
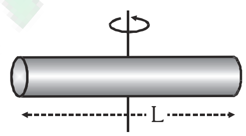
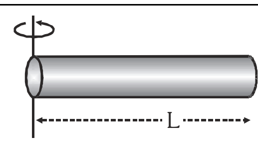
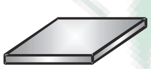
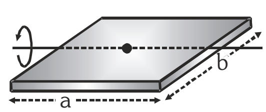
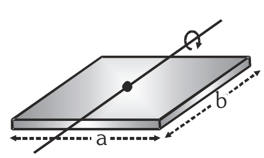
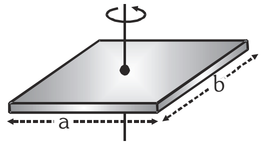
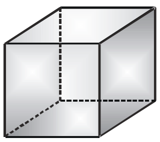
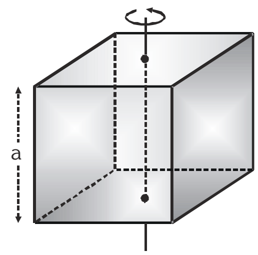
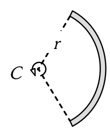
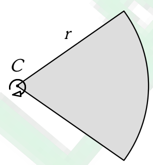
MOMENT OF INERTIA OF SOME REGULAR BODIES

Shape of the body	**Position of the axis of rotation**	**Figure**	**Moment of Inertia (I)**	**Radius of gyration (K)**
(1) Circular Ring  Mass = M Radius = R	(a) About an axis perpendicular to the plane and passes through the centre		MR^2	R
	(b) About the diametric axis		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
	(c) About an axis tangential to the rim and perpendicular to the plane of the ring		$2MR^2$	$\sqrt{2}R$
	(d) About an axis tangential to the rim and lying in the plane of ring		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$
(2) Circular Disc  M = Mass R = Radius	(a) About an axis passing through the centre and perpendicular to the plane of disc		$\frac{1}{2}MR^2$	$\frac{R}{\sqrt{2}}$
	(b) About a diametric axis		$\frac{MR^2}{4}$	$\frac{R}{2}$
	(c) About an axis tangential to the rim and lying in the plane of the disc		$\frac{5}{4}MR^2$	$\frac{\sqrt{5}}{2}R$
	(d) About an axis tangential to the rim & perpendicular to the plane of disc		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$
(3) Annular disc  M = Mass R_1 = Internal Radius R_2 = Outer Radius	(a) About an axis passing through the centre and perpendicular to the plane of disc		$\frac{M}{2}\left[R_1^2 + R_2^2\right]$	$\sqrt{\frac{R_1^2 + R_2^2}{2}}$

Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration (K)
	(b) About a diametric axis		$\frac{M}{4}\left[R_1^2+R_2^2\right]$	$\frac{\sqrt{R_1^2+R_2^2}}{2}$
(4) Solid Sphere  M = Mass R = Radius	(a) About its diametric axis which passes through its centre of mass		$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$
	(b) About a tangent to the Sphere		$\frac{7}{5}MR^2$	$\sqrt{\frac{7}{5}}R$
(5) Hollow Sphere (Thin spherical Shell) 	(a) About diametric axis passing through centre of mass	 table tennis ball	$\frac{2}{3}MR^2$	$\sqrt{\frac{2}{3}}R$
M = Mass R = Radius Thickness negligible	(b) About a tangent to the surface	 table tennis ball	$\frac{5}{3}MR^2$	$\sqrt{\frac{5}{3}}R$
(6) Hollow Cylinder 	(a) About its geometrical axis which is parallel to its length		MR^2	R
M = Mass R = Radius L = Length	(b) About an axis which is perpendicular to its length and passes through its centre of mass		$\frac{MR^2}{2}+\frac{ML^2}{12}$	$\sqrt{\frac{R^2}{2}+\frac{L^2}{12}}$

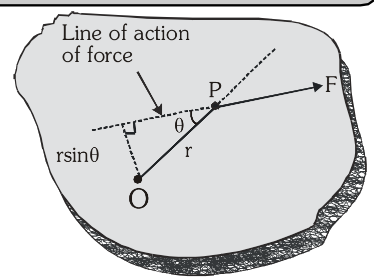
Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration (K)
	(c) About an axis perpendicular to its length and passing through one end of the cylinder		$\frac{MR^2}{2}+\frac{ML^2}{3}$	$\sqrt{\frac{R^2}{2}+\frac{L^2}{3}}$
(7) Solid Cylinder M = Mass R = Radius L = Length 	(a) About its geometrical axis, which is along its length		$\frac{MR^2}{2}$	$\frac{R}{\sqrt{2}}$
	(b) About an axis tangential to the cylinderical surface and parallel to its geometrical axis		$\frac{3}{2}MR^2$	$\sqrt{\frac{3}{2}}R$
	(c) About an axis passing through the centre of mass and perpendicular to its length		$\frac{ML^2}{12}+\frac{MR^2}{4}$	$\sqrt{\frac{L^2}{12}+\frac{R^2}{4}}$
(8) Thin Rod  Thickness is negligible w.r.t. length	(a) About an axis passing through centre of mass and perpendicular to its length		$\frac{ML^2}{12}$	$\frac{L}{\sqrt{12}}$
Mass = M Length = L	(b) About an axis passing through one end and perpendicular to length of the rod		$\frac{ML^2}{3}$	$\frac{L}{\sqrt{3}}$
(9) Rectangular Plate 	(a) About an axis passing through centre of mass and perpendicular to side b in its plane		$\frac{Mb^2}{12}$	$\frac{b}{2\sqrt{3}}$
M = Mass a = Length b = Breadth	(b) About an axis passing through centre of mass and perpendicular to side a in its plane.		$\frac{Ma^2}{12}$	$\frac{a}{2\sqrt{3}}$

Shape of the body	Position of the axis of rotation	Figure	Moment of Inertia (I)	Radius of gyration (K)
	(c) About an axis passing throught centre of mass and perpendicular to plane		$\frac{M\left(a^2+b^2\right)}{12}$	$\sqrt{\frac{a^2+b^2}{12}}$
(10) Cube  Mass = M Side a	About an axis passes through centre of mass and perpendicular to face		$\frac{Ma^2}{6}$	$\frac{a}{\sqrt{6}}$
(11) Uniform thin rod bent into shape of an arc of mass m	About an axis Passing through center and perpendicular to the plane containing the arc		$I_C = mr^2$	r
(12) Sector of a uniform disk of mass m	About an axis Passing through center and perpendicular to the plane containing the sector.		$I_C = \frac{mr^2}{2}$	$r/\sqrt{2}$

TORQUE

Torque about point : $\vec{\tau} = \vec{r} \times \vec{F}$

Magnitude of torque = Force × perpendicular distance of line of action of force from the axis of rotation.

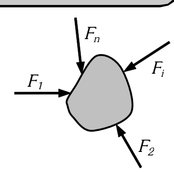


$$t = rF\sin\theta$$

Direction of torque can be determined by using right hand thumb rule.

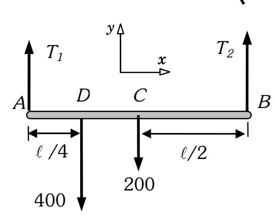
ROTATIONAL EQUILIBRIUM

If a rigid body is in rotational equilibrium under the action of several coplanar forces, the resultant torque of all the forces about any axis perpendicular to the plane containing the forces must be zero.



In the figure a body is shown under the action of several external coplanar forces F_1, F_2, ... F_i and F_n.

$$\sum \vec{\tau}_P = 0$$



Here P is a point in the plane of the forces about which we calculate torque of all the external forces acting on the body. The flexibility available in selection of the point P provides us with advantages that we can select such a point about which torques of several unknown forces will become zero or we can make as many number of equations as desired by selecting several different points. The first situation yields to a simpler equation to be solved and second situation though does not give independent equation, which can be used to determine additional unknowns yet may be used to check the solution.

The above condition reveals that a body cannot be in rotational equilibrium under the action of a single force unless the line of action passes through the mass center of the body.

A case of particular interest arises where only three coplanar forces are involved and the body is in rotational equilibrium. It can be shown that *if a body is in rotational equilibrium under the action of three forces, the lines of action of the three forces must be either concurrent or parallel.* This condition provides us with a graphical technique to analyze rotational equilibrium.

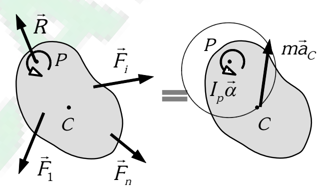
Equilibrium of Rigid Bodies

A rigid body is said to be in equilibrium, if it is in translational as well as rotational equilibrium both. To analyze such problems conditions for both the equilibriums must be applied.

Rotation about fixed axis not passing through mass center

In this kind of rotation the axis of rotation remains fixed and does not passes through the mass center. Rotation of door is a common example of this category. Doors are hinged about their edges; therefore their axis of rotation does not pass through the mass center. In this kind of rotation motion the mass center executes circular motion about the axis of rotation.

In the figure, free body diagram and kinetic diagram of a body rotating about a fixed axis through point P is shown. It is easy to conceive that as the body rotates its mass center moves on a circular path of radius $\vec{r}_{P/C}$. The mass center of the body is in translation motion with acceleration $\vec{a}_C$ on circular path of radius $r_{P/C}$. To deal with this kind of motion, we have to make use of both the force and the torque equations.



Translation of mass center $\quad \Sigma\vec{F}_i = M\vec{a}_C = M\vec{\alpha}\times\vec{r}_{C/P} - M\omega^2\vec{r}_{C/P}$

Centroidal Rotation $\quad \Sigma\vec{\tau}_C = I_C\vec{\alpha}$

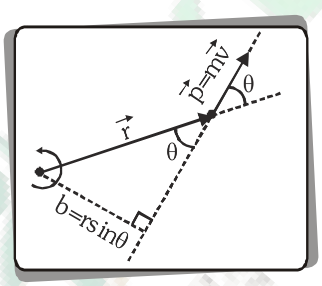
Making use of parallel axis theorem $\quad (I_P = Mr^2_{P/C} + I_C)$ and $\vec{a}_{C/P} = \vec{\alpha}\times\vec{r}_{C/P} - \omega^2\vec{r}_{C/P}$

we can write the following equation also.

Pure Rotation about P $\quad \Sigma\vec{\tau}_P = I_P\vec{\alpha}$

ANGULAR MOMENTUM (MOMENT OF LINEAR MOMENTUM)

Angular momentum of a particle about a given axis is the product of its linear momentum and perpendicular distance of line of action of linear momentum vector from the axis of rotation, $\vec{L} = \vec{r}\times\vec{p}$



Magnitude of Angular momentum

= Linear momentum × Perpendicular distance of line of action of momentum from the axis of rotation

$L = mv \times r \sin\theta$

Direction of angular momentum can be used by using right hand thumb rule.

- According to Newtons Second Law's for rotatory motion $\vec{\tau} = \frac{d\vec{L}}{dt} = I\vec{\alpha}$.
- Angular Impulse = Change in angular momentum.
- If a large torque acts on a body for a small time then, angular impulse= $\vec{\tau}dt$

Conservation of Angular Momentum

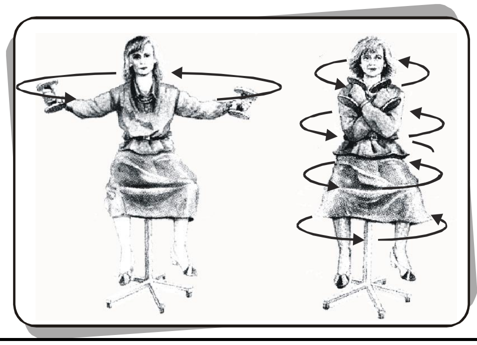
Angular momentum of a particle or a system remains constant if $\tau_{ext} = 0$ about that point or axis of rotation.

If $\tau = 0$ then $\frac{\Delta L}{\Delta t} = 0 \Rightarrow L =$ constant

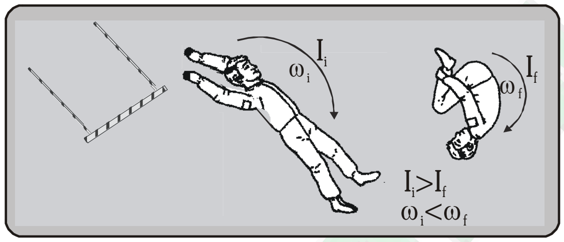
$\Rightarrow L_f = L_i$ or $I_1\omega_1 = I_2\omega_2$

Examples of Conservation of Angular Momentum

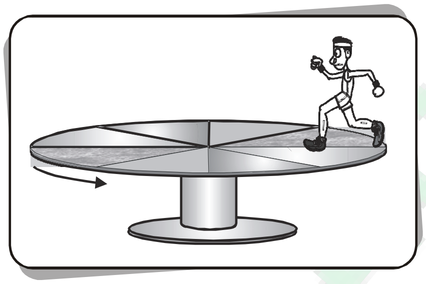
- If a person skating on ice folds his arms then his M.I. decreases and 'ω' increases.



- A diver jumping from a height folds his arms and legs (I decrease) in order to increase no. of rotation in air by increasing 'ω'.



- If a person moves towards the centre of rotating platform then 'I' decrease and 'ω' increase.



ROTATIONAL KINETIC ENERGY

Kinetic Energy of Rotation $KE_R = \frac{1}{2}I\omega^2$

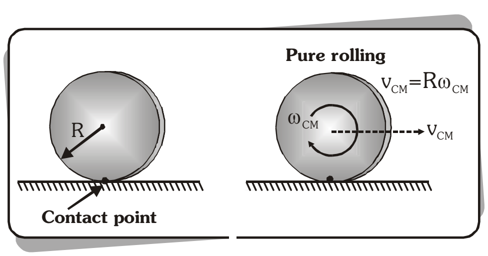
- Other forms $K = \frac{1}{2}I\omega^2 = \frac{L^2}{2I} = \frac{1}{2}L\omega$
- If external torque acting on a body is equal to zero ($\tau = 0$), L=constant $K \propto \frac{1}{I}$, $K \propto \omega$
- Rotational Work : $W_r = \tau\theta$ (If torque is constant) $W_r = \int_{\theta_1}^{\theta_2} \tau d\theta$ (If torque is variable)
- The work done by torque = Change in kinetic energy of rotation. $W = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2 = \frac{1}{2}I(\omega_2^2 - \omega_1^2)$
- Instantaneous power $= \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$ Average power $P_{av} = \frac{\Delta W}{\Delta t}$

COMBINED TRANSLATIONAL AND ROTATIONAL MOTION OF A RIGID BODY

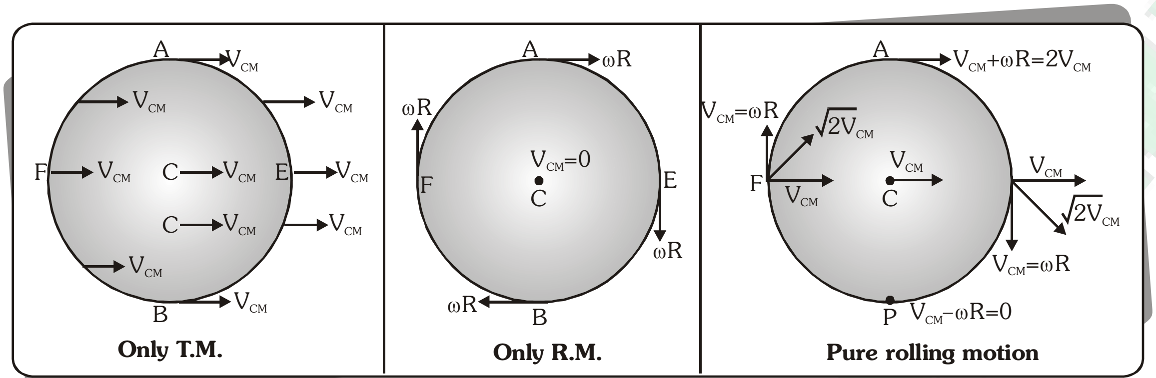
When a body perform translatory motion as well as rotatory motion then it is known as rolling.

In Pure Rolling

(i) If the velocity of point of contact with respect to the surface is zero then it is known as pure rolling.



(ii) If a body is performing rolling then the velocity of any point of the body with respect to the surface is given by $\vec{v} = \vec{v}_{CM} + \vec{\omega}_{CM} \times \vec{R}$

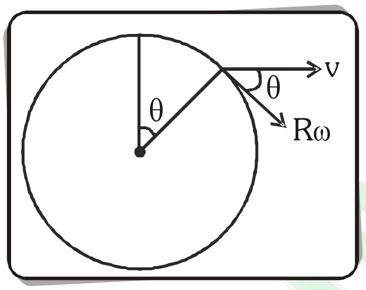


Only Translatory motion + Only Rotatory Motion = Rolling motion. For pure rolling above body

$V_A = 2V_{CM}$ $\quad V_E = \sqrt{2}\, V_{CM}$ $\quad V_F = \sqrt{2}\, V_{CM}$ $\quad V_B = 0$

Velocity at a point on rim of sphere

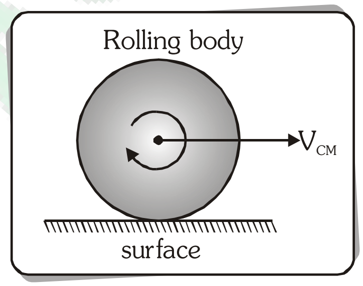
$$v_{net} = \sqrt{v^2 + R^2\omega^2 + 2vR\omega\cos\theta}$$



For pure rolling $v = R\omega$

$$v_{net} = 2v\cos\frac{\theta}{2}$$

Rolling Kinetic Energy under pure rolling



Rolling Kinetic Energy

$$E = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}mK^2\left(\frac{v^2}{R^2}\right)$$

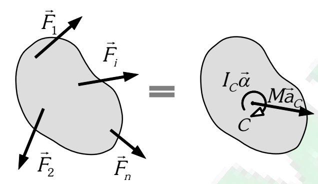
Rolling Kinetic Energy $E = \frac{1}{2}mv^2\left(1 + \frac{K^2}{R^2}\right)$

$$E_{translation} : E_{rotation} : E_{Total} = 1 : \frac{K^2}{R^2} : 1 + \frac{K^2}{R^2}$$

Body	$\frac{K^2}{R^2}$	$\frac{E_{trans}}{E_{rotation}} = \frac{1}{\frac{K^2}{R^2}}$	$\frac{E_{trans}}{E_{total}} = \frac{1}{1+\frac{K^2}{R^2}}$	$\frac{E_{rotation}}{E_{total}} = \frac{\frac{K^2}{R^2}}{1+\frac{K^2}{R^2}}$
Ring	1	1	1/2	1/2
Disc	1/2	2	2/3	1/3
Solid sphere	2/5	5/2	5/7	2/7
Spherical shell	2/3	3/2	3/5	2/5
Solid cylinder	1/2	2	2/3	1/3
Hollow cylinder	1	1	1/2	1/2

General Plane Motion: Rotation about axis in translation motion

Rotation of bodies about an axis in translation motion can be dealt with either as superposition of translation of mass center and centroidal rotation or assuming pure rotation about the instantaneous axis of rotation. In the figure is shown the free body diagram and kinetic diagram of a body in general plane motion.

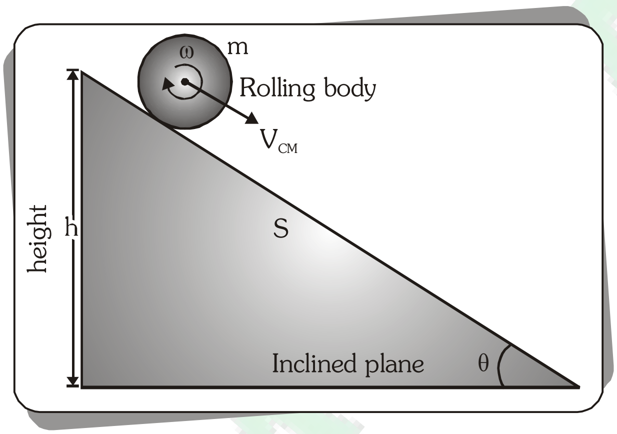


Translation of mass center $\sum_{i=0}^{n} \vec{F}_i = M\vec{a}_C$

Centroidal Rotation $\sum_{i-1}^{n} \vec{\tau}_C = I_C\vec{\alpha}$

This kind of situation can also be dealt with considering it rotation about IAR. It gives sometimes quick solutions, especially when IAR is known and forces if acting at the IAR are not required to be found.

Rolling Motion on an inclined plane



Applying Conservation of energy

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}mK^2\left(\frac{v^2}{R^2}\right)$$

$$mgh = \frac{1}{2}mv^2\left(1 + \frac{K^2}{R^2}\right) \quad ...(1)$$

$$h = s\sin\theta \quad ...(2)$$

from (1) & (2)

$$V_{Rolling} = \sqrt{\frac{2gh}{1+\frac{K^2}{R^2}}} = \sqrt{\frac{2gs\sin\theta}{1+\frac{K^2}{R^2}}}$$

- Linear accleration on reaching the lowest point a

$$= \frac{g\sin\theta}{1+K^2/R^2}$$

- Time taken to reach the lowest point of the plane is

$$t = \sqrt{\frac{2s(1+K^2/R^2)}{g\sin\theta}}$$

- $\frac{K^2}{R^2}$ Least, will reach first

  $\frac{K^2}{R^2}$ Maximum, will reach last

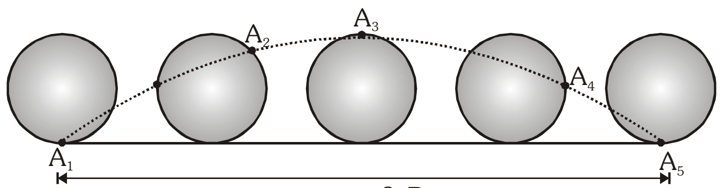
  $\frac{K^2}{R^2}$ equal, will reach together

- When ring, disc, hollows sphere, solid sphere rolls on same inclined plane then

  $v_S > v_D > v_H > v_R$ $\quad a_S > a_D > a_H > a_R$

  $t_S < t_D < t_H < t_R$

For a pure rolling body after one full rotation

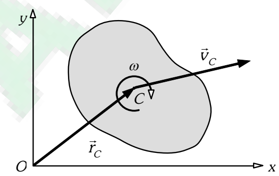


displacement of lowermost point $= 2\pi R$

distance = 8R

Angular Momentum in general plane motion

Angular momentum of a body in plane motion can also be written similar to torque equation or kinetic energy as sum of angular momentum about the axis due to translation of mass center and angular momentum of centroidal rotation about centroidal axis parallel to the original axis.



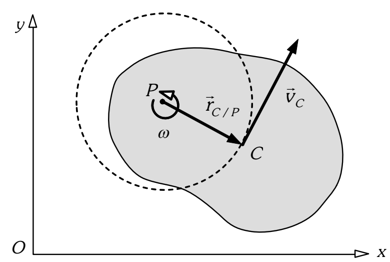
Consider a rigid body of mass M in plane motion. At the instant shown its mass center has velocity $\vec{v}$ and it is rotating with angular velocity $\vec{\omega}$ about an axis perpendicular to the plane of the figure. It angular momentum $\vec{L}_o$ about an axis passing though the origin and parallel to the original is expressed by the following equation.

$$\vec{L}_o = \vec{r}_C \times \left(M\vec{v}_C\right) + I_C\vec{\omega}$$

The first term of the above equation represent angular momentum due to translation of the mass center and the second term represents angular momentum in centroidal rotation.

Angular momentum in rotation about fixed axis

Consider a body of mass M rotating with angular velocity ω about a fixed axis perpendicular to plane of the figure passing through point P. Making use of the parallel axis theorem $I_P = Mr_{C/P}^2 + I_C$ and equation $\vec{v}_C = \vec{\omega} \times \vec{r}_{C/P}$ we can express the angular momentum $\vec{L}_P$ of the body about the fixed rotational axis.

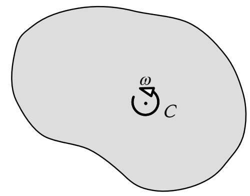


$$\vec{L}_P = I_P\vec{\omega}$$

The above equation reveals that the angular momentum of a rigid body in plane motion can also be expressed in a single term due to rotation about the instantaneous axis of rotation.

Angular momentum in pure centroidal rotation

In pure centroidal rotation, mass center remains at rest, therefore angular momentum due to translation of the mass center vanishes.

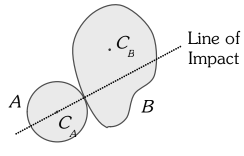


$$\vec{L}_C = I_C\vec{\omega}$$

Eccentric Impact

In eccentric impact the line of impact which is the common normal drawn at the point of impact does not passes through mass center of at least one of the colliding bodies. It involves change in state of rotation motion of either or both the bodies.

Consider impact of two A and B such that the mass center C_B of B does not lie on the line of impact as shown in figure. If we assume bodies to be frictionless their mutual forces must act along the line of impact. The reaction force of A on B does not passes through the mass center of B as a result state of rotation motion of B changes during the impact.



Problems of Eccentric Impact

Problems of eccentric impact can be divided into two categories. In one category both the bodies under going eccentric impact are free to move. No external force act on either of them. There mutual forces are responsible for change in their momentum and angular momentum. In another category either or both of the bodies are hinged.

Eccentric Impact of bodies free to move

Since no external force acts on the two body system, we can use principle of conservation of linear momentum, principle of conservation of angular momentum about any point and concept of coefficient of restitution.

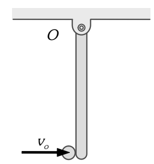
The coefficient of restitution is defined for components of velocities of points of contacts of the bodies along the line of impact.

While applying principle of conservation of angular momentum care must be taken in selecting the point about which we write the equation. The point about which we write angular momentum must be at rest relative to the selected inertial reference frame and as far as possible its location should be selected on line of velocity of the mass center in order to make zero the first term involving moment of momentum of mass center.

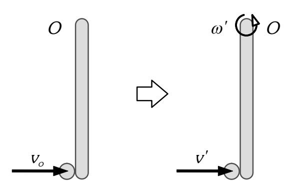
Eccentric Impact of hinged bodies

When either or both of the bodies are hinged the reaction of the hinge during the impact act as external force on the two body system, therefore linear momentum no longer remain conserved and we cannot apply principle of conservation of linear momentum. When both the bodies are hinged we cannot also apply conservation of angular momentum, and we have to use impulse momentum principle on both the bodies separately in addition to making use of coefficient of restitution. But when one of the bodies is hinged and other one is free to move, we can apply conservation of angular momentum about the hinge.

Ex. A uniform rod of mass m and length ℓ is suspended from a fixed support and can rotate freely in the vertical plane. A small ball of mass m moving horizontally with velocity v_o strikes elastically the lower end of the rod as shown in the figure. Find the angular velocity of the rod and velocity of the ball immediately after the impact.



Sol. The rod is hinged and the ball is free to move. External forces acting on the rod ball system are their weights and reaction from the hinge. Weight of the ball as well as the rod are finite and contribute negligible impulse during the impact, but impulse of reaction of the hinge during impact is considerable and cannot be neglected. Obviously linear momentum of the system is not conserved. The angular impulse of the reaction of hinge about the hinge is zero. Therefore angular momentum of the system about the hinge is conserved. Let velocity of the ball after the impact becomes v'_B and angular velocity of the rod becomes ω'.



Before the impact   Immediately after the impact

We denote angular momentum of the ball and the rod about the hinge before the impact by L_{B1} and L_{R1} and after the impact by L_{B2} and L_{R2}.

Applying conservation of angular momentum about the hinge, we have

$$\vec{L}_{B1} + \vec{L}_{R1} = \vec{L}_{B2} + \vec{L}_{R2} \rightarrow \quad mv_o\ell + 0 = mv'_B\ell + I_o\omega'$$

Substituting $\frac{1}{3}M\ell^2$ for I_o, we have

$$3mv'_B + M\ell\omega' = 3mv_o \qquad (1)$$

The velocity of the lower end of the rod before the impact was zero and immediately after the impact it becomes $\ell\omega'$ towards right. Employing these facts we can express the coefficient of restitution according to eq.

$$e = \frac{v'_{Qn} - v'_{Pn}}{v_{pn} - v_{Qn}} \rightarrow \qquad \ell\omega' - v'_B = ev_o \qquad (2)$$

From eq. (1) and (2), we have

Velocity of the ball immediately after the impact $\quad v'_B = \dfrac{(3m - eM)v_o}{3m + M}$ **Ans.**

Angular velocity of the rod immediately after the impact $\quad \omega' = \dfrac{3(1+e)mv_o}{(3m+M)\ell}$ **Ans.**