WAVE MOTION & DOPPLER'S EFFECT

A wave is a disturbance that propagates in space, transports energy and momentum from one point to another without the transport of matter.

CLASSIFICATION OF WAVES

- Medium Necessity
  - Mechanical (Elastic) waves
  - Non-mechanical waves (EM waves)
- Propagation of energy
  - Progressive waves
  - Stationary (standing) waves
- Vibration of medium particle
  - Transverse waves
  - Longitudinal waves
- Dimension
  - 1-D (Waves on strings)
  - 2-D (Surface waves or ripples on water)
  - 3-D (Sound or light waves)

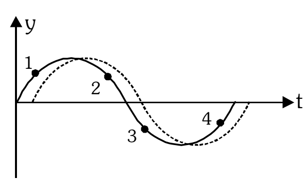
- If particle vibrates in pependiuclar direction to the wave motion then its called transverse wave.
- In strings, mechanical waves are always transverse.
- In gases and liquids, mechanical waves are always longitudinal because fluids cannot sustain shear.
- Partially transverse waves are possible on a liquid surface because surface tension provide some rigidity on a liquid surface. These waves are called as ripples as they are combination of transverse & longitudinal.
- In solids mechanical waves (may be sound) can be either transverse or longitudinal depending on the mode of excitation.
- In longitudinal wave motion, oscillatory motion of the medium particles produce regions of compression (high pressure) and rarefaction (low pressure).

PLANE PROGRESSIVE WAVES

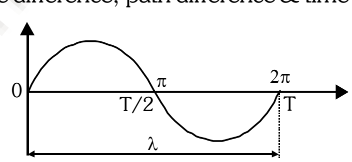
- Wave equation : $y = A\sin(\omega t - kx)$ where $k = \frac{2\pi}{\lambda}$ = wave propagation constant
  (Wave is moving along x-axis and paticle is moving along y axis)
- If coefficient of x & t are of opposite sign than wave move in +ve x-direction
  Ex. $y = A\sin(\omega t - kx)$ wave is moving in +x direction and particle is moving in y direction
  Similarly Ex. $z = A\sin(\omega t - ky)$ here wave move in +y direction and particle move in z direction
- Differential equation : $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$

  Wave velocity (phase velocity) $\boxed{v_w = \frac{dx}{dt} = \frac{\omega}{k}}$ $\because \omega t - kx = \text{constant} \Rightarrow \frac{dx}{dt} = \frac{\omega}{k}$
- Particle velocity $v_p = \frac{dy}{dt} = A\omega\cos(\omega t - kx)$ $\boxed{v_p = -v_w \times \text{slope} = -v_w\left(\frac{dy}{dx}\right)}$
- Particle acceleration : $a_p = \frac{\partial^2 y}{\partial t^2} = -\omega^2 A\sin(\omega t - kx) = -\omega^2 y$

  For particle 1 : $v_p \downarrow$ and $a_p \downarrow$
  For particle 2 : $v_p \uparrow$ and $a_p \downarrow$
  For particle 3 : $v_p \uparrow$ and $a_p \uparrow$
  For particle 4 : $v_p \downarrow$ and $a_p \uparrow$



- Relation between phase difference, path difference & time difference



$$\boxed{\frac{\Delta\phi}{2\pi} = \frac{\Delta\lambda}{\lambda} = \frac{\Delta T}{T}}$$

ENERGY IN WAVE MOTION

- $$\frac{KE}{volume} = \frac{1}{2}\left(\frac{\Delta m}{volume}\right) v_p^2 = \frac{1}{2}\rho v_p^2 = \frac{1}{2}\rho\omega^2 A^2 \cos^2(\omega t - kx)$$
- $$\frac{PE}{volume} = \frac{1}{2}\rho v^2 \left(\frac{dy}{dx}\right)^2 = \frac{1}{2}\rho\omega^2 A^2 \cos^2(\omega t - kx)$$
- $$\frac{TE}{volume} = \rho\omega^2 A^2 \cos^2(\omega t - kx)$$
- Pressure energy density $\boxed{u = \frac{1}{2}\rho\omega^2 A^2}$

  [i.e. Average total energy / volume]
- **Power :** P = (energy density) (volume/ time)

  $$P = \left(\frac{1}{2}\rho\omega^2 A^2\right)(Sv)$$

  [where S = Area of cross-section]
- **Intensity :** $I = \frac{Power}{\text{area of cross-section}} = \frac{1}{2}\rho\omega^2 A^2 v$

Speed of transverse wave on string :

$v = \sqrt{\frac{T}{\mu}}$ where μ = mass/length and

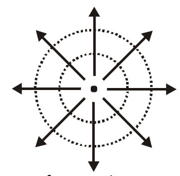
T = tension in the string.

KEY POINTS

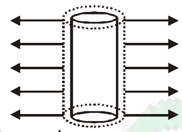
- A wave can be represented by function $y=f(kx \pm \omega t)$ because it satisfy the differential equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\left(\frac{\partial^2 y}{\partial t^2}\right)$ where $v = \frac{\omega}{k}$.
- A pulse whose wave function is given by $y=4 / [(2x+5t)^2+2]$ propagates in –x direction as this wave function is of the form $y=f(kx + \omega t)$ which represent a wave travelling in –x direction.
- Longitudinal waves can be produced in solids, liquids and gases because bulk modulus of elasticity is present in all three.

WAVE FRONT

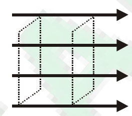
- Spherical wave front (source →point source)



- Cylindrical wave front (source →linear source)



- Plane wave front (source → point / linear source at very large distance)



INTENSITY OF WAVE

- Due to point source $I \propto \frac{1}{r^2}$

  $$y(r,t) = \frac{A}{r}\sin(\omega t - \vec{k}\cdot\vec{r})$$
- Due to cylindrical source $I \propto \frac{1}{r}$

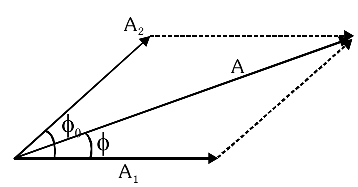
  $$y(r,t) = \frac{A}{\sqrt{r}}\sin(\omega t - \vec{k}\cdot\vec{r})$$
- Due to plane source I = constant

  $$y(r,t) = A\sin(\omega t - \vec{k}\cdot\vec{r})$$

INTERFERENCE OF WAVES

$y_1 = A_1\sin(\omega t - kx)$, $y_2 = A_2\sin(\omega t - kx + \phi_0)$

$y = y_1 + y_2 = A \sin(\omega t - kx + \phi)$



where $A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos\phi_0}$

and $\tan\phi = \frac{A_2\sin\phi_0}{A_1 + A_2\cos\phi_0}$

As $I \propto A^2$

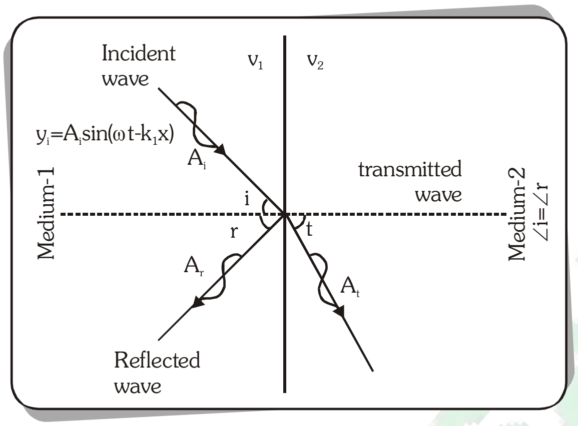
So $\boxed{I = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\phi_0}$

- For constructive interference [Maximum intensity] $\phi_0 = 2n\pi$ or path difference $= n\lambda$ where n = 0, 1, 2, 3, ... $I_{max} = \left(\sqrt{I_1} + \sqrt{I_2}\right)^2$
- For destructive interference [Minimum Intensity]

  $\phi_0 = (2n+1)\pi$ or path difference $= (2n+1)\frac{\lambda}{2}$

  where n = 0, 1, 2, 3, ... $I_{min} = \left(\sqrt{I_1} - \sqrt{I_2}\right)^2$
- Degree of hearing$= \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \times 100$

REFLECTION AND REFRACTION (TRANSMISSION) OF WAVES

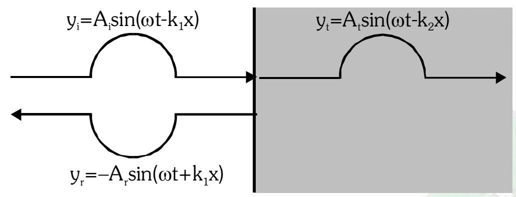


- The frequency of the wave remain unchanged.
- Amplitude of reflected wave→ $A_r = \left(\frac{v_2 - v_1}{v_1 + v_2}\right)A_i$
- Amplitude of transmitted wave → $A_t = \left(\frac{2v_2}{v_1 + v_2}\right)A_i$
- If $v_2 > v_1$ i.e. medium-2 is rarer

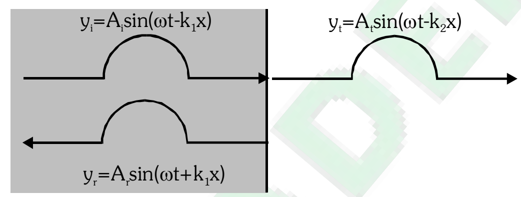
  $A_r > 0 \Rightarrow$ no phase change in reflected wave
- If $v_2 < v_1$ i.e. medium-1 is rarer

  $A_r < 0 \Rightarrow$ There is a phase change of π in reflected wave
- As A_t is always positive whatever be v_1 & v_2 the phase of transmitted wave always remains unchanged.
- In case of reflection from a denser medium or rigid support or fixed end, there is inversion of reflected wave i.e. phase difference of π between reflected and incident wave.
- The transmitted wave is never inverted.

- **Rarer** **Denser**

- **Denser** **Rarer**

BEATS

When two sound waves of nearly equal (but not exactly equal) frequencies travel in same direction, at a given point due to their super position, intensity alternatively increases and decreases periodically. This periodic waxing and waning of sound at a given position is called beats. Beat frequency = difference of frequencies of two interfering waves

Beat frequency = $|f_1 - f_2|$

STATIONARY WAVES OR STANDING WAVES

When two waves of same frequency and amplitude travel in opposite direction at same speed, their superstition gives rise to a new type of wave, called stationary waves or standing waves. Formation of standing wave is possible only in bounded medium.

- Let two waves are y_1=Asin(ωt–kx); y_2=Asin(ωt+kx) by principle of superposition $y=y_1+y_2$ =2Acoskxsinωt ← Equation of stationary wave
- As this equation satisfies the wave equation

  $\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$, it represent a wave.
- Its amplitude is not constant but varies periodically with position.
- **Nodes**→amplitude is minimum :

  $\cos kx = 0 \Rightarrow x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \frac{5\lambda}{4}, ...$
- **Antinodes** → amplitude is maximum :

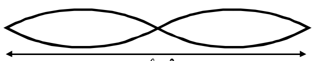
  $\cos kx = 1 \Rightarrow x = 0, \frac{\lambda}{2}, \lambda, \frac{3\lambda}{2}, ...$
- The nodes divide the medium into segments (loops). All the particles in a segment vibrate in same phase but in opposite phase with the particles in the adjacent segment.
- As nodes are permanently at rest, so no energy can be transmitted across them, i.e. energy of one region (segment) is confined in that region.

Transverse stationary waves in stretched string

[Fixed at both ends] [fixed end →Node & free end→Antinode]


$\ell = \frac{\lambda}{2}$

Fundamental or first harmonic or zero overtone $f = \frac{v}{2\ell}$


$\ell = \lambda$

second harmonic first overtone $f = \frac{2v}{2\ell}$


$\ell = \frac{3\lambda}{2}$

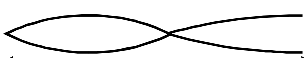
third harmonic second overtone $f = \frac{3v}{2\ell}$

n^{th} harmonic $(n-1)^{th}$ overtone $f_\ell = \frac{nv}{2\ell}$

Fixed at one end


$\ell = \frac{\lambda}{4}$

Fundamental $f = \frac{v}{4\ell}$


$\ell = \frac{3\lambda}{4}$

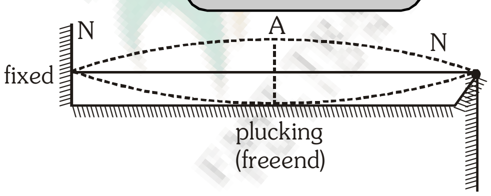
third harmonic first overtone $f = \frac{3v}{4\ell}$


$\ell = \frac{5\lambda}{4}$

fifth harmonic second overtone $f = \frac{5v}{4\ell}$

$(2n+1)^{th}$ harmonic n^{th} overtone $f = \frac{(2n+1)v}{4\ell}$

Sonometer



$f_n = \frac{p}{2\ell}\sqrt{\frac{T}{\mu}}$

[p : number of loops]

Sound Waves

Velocity of sound in a medium of elasticity E and density ρ is

$$v = \sqrt{\frac{E}{\rho}}$$

Solids (Young's Modulus): $v = \sqrt{\frac{Y}{\rho}}$

Fluids (Bulk Modulus): $v = \sqrt{\frac{B}{\rho}}$

- **Newton's formula** : Sound propagation is isothermal $B = P \Rightarrow v = \sqrt{\frac{P}{\rho}}$
- **Laplace correction** : Sound propagation is adiabatic $B = \gamma P \Rightarrow v = \sqrt{\frac{\gamma P}{\rho}}$

KEY POINTS

- With rise in temperature, velocity of sound in a gas increases as $v = \sqrt{\frac{\gamma RT}{M_W}}$
- With rise in humidity velocity of sound increases due to presence of water in air.
- Pressure has no effect on velocity of sound in a gas as long as temperature remains constant.

Displacement and pressure wave

A sound wave can be described either in terms of the longitudinal displacement suffered by the particles of the medium (called displacement wave) or in terms of the excess pressure generated due to compression and rarefaction (called pressure wave).

Displacement wave	$y = A\sin(\omega t - kx)$
Pressure wave	$p = p_0\cos(\omega t - kx)$
where	$p_0 = ABk = \rho A v \omega$

Note : As sound-sensors (e.g., ear or mike) detect pressure changes, description of sound as pressure wave is preferred over displacement wave.

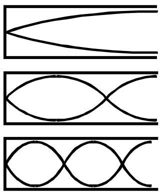
KEYPOINTS

- The pressure wave is 90° out of phase w.r.t. displacement wave, i.e. displacement will be maximum when pressure is minimum and vice-versa.
- Intensity in terms of pressure amplitude $I = \frac{p_0^2}{2\rho v}$

Vibrations of organ pipes

Stationary longitudinal waves closed end → displacement node, open end→ displacement antinode

- **Closed end organ pipe**



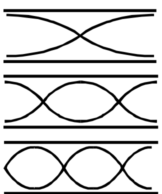
$\ell = \frac{\lambda}{4} \Rightarrow f = \frac{v}{4\ell}$

$\ell = \frac{3\lambda}{4} \Rightarrow f = \frac{3v}{4\ell}$

$\ell = \frac{5\lambda}{4} \Rightarrow f = \frac{5v}{4\ell}$

  - Only odd harmonics are present
  - Maximum possible wavelength = 4ℓ
  - Frequency of m^{th} overtone $= (2m+1)\frac{v}{4\ell}$

- **Open end organ pipe**



$\ell = \frac{\lambda}{2} \Rightarrow f = \frac{v}{2\ell}$

$\ell = \lambda \Rightarrow f = \frac{2v}{2\ell}$

$\ell = \frac{3\lambda}{2} \Rightarrow f = \frac{3v}{2\ell}$

  - All harmonics are present
  - Maximum possible wavelength is 2ℓ.
  - Frequency of m^{th} overtone $= (m+1)\frac{v}{2\ell}$

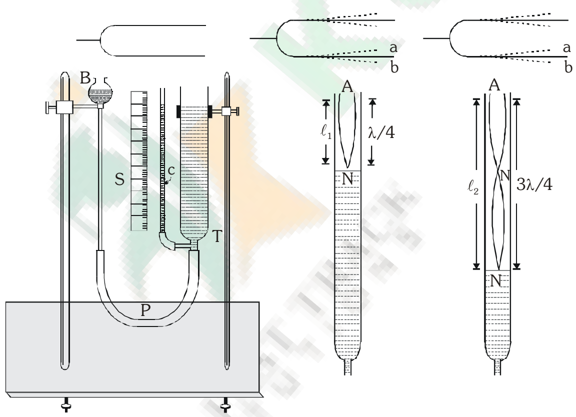
- **End correction** :
  Due to finite momentum of air molecules in organ pipes reflection takes place not exactly at open end but some what above it, so antinode is not formed exactly at free end but slightly above it.

  In closed organ pipe $f_1 = \frac{v}{4(\ell + e)}$

  where e = 0.6 R (R=radius of pipe)

  In open organ pipe $f_1 = \frac{v}{2(\ell + 2e)}$

- **Resonance Tube**



Wavelength $\lambda = 2(\ell_2 - \ell_1)$

End correction $e = \frac{\ell_2 - 3\ell_1}{2}$

Intensity of sound in decibels

Sound level, $SL = 10\log_{10}\left(\frac{I}{I_0}\right)$

Where I_0 = threshold of human ear = 10^{-12} W/m²

Characteristics of sound

- Loudness → Sensation received by the ear due to intensity of sound.
- Pitch → Sensation received by the ear due to frequency of sound.
- Quality (or Timbre)→ Sensation received by the ear due to waveform of sound.

Doppler's effect in sound :

A stationary source emits wave fronts that propagate with constant velocity with constant separation between them and a stationary observer encounters them at regular constant intervals at which they were emitted by the source.

A moving observer will encounter more or lesser number of wavefronts depending on whether he is approaching or receding the source.

A source in motion will emit different wave front at different places and therefore alter wavelength i.e. separation between the wavefronts.

The apparent change in frequency or pitch due to relative motion of source and observer along the line of sight is called Doppler Effect.

Observed frequency

$$n' = \frac{\text{speed of sound wave w.r.t. observer}}{\text{observed wavelength}}$$

$$n' = \frac{v + v_0}{\left(\frac{v - v_s}{n}\right)} = \left(\frac{v + v_0}{v - v_s}\right)n$$

If v_0, v_s <<<v then $n' \approx \left(1 + \frac{v_0 + v_s}{v}\right)n$

- Mach Number$= \frac{\text{speed of source}}{\text{speed of sound}}$

Doppler's effect in light :

Case I : **Observer** O ●   **Light Source** S ● ← v

Frequency $v' = \left(\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\right)v \approx \left(1 + \frac{v}{c}\right)v$

Wavelength $\lambda' = \left(\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\right)\lambda \approx \left(1 - \frac{v}{c}\right)\lambda$

} Violet Shift

Case II : **Observer** O ●   **Light Source** S ● → v

Frequency $v' = \left(\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}\right)v \approx \left(1 - \frac{v}{c}\right)v$

Wavelength $\lambda' = \left(\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\right)\lambda \approx \left(1 + \frac{v}{c}\right)\lambda$

} Red Shift